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THE DETACHMENT OF AN ELASTIC STRIP CONTAINING ABSOLUTELY RIGID INCLUSIONS FROM A SUPPORT†

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A problem of the theory of elasticity for a strip containing rigid inclusions is considered. The faces of the strip are supported without friction by a rigid base and the inclusions are loaded by forces perpendicular to the faces of the strip, which leads to the appearance of a section of the strip detached from one of the supporting surfaces, in which case the problem becomes a strictly mixed one. We use the method developed in [1] for non-mixed problems in the case of non-canonical domains. It is based on the construction of matrix-valued Green's functions corresponding to simpler classical boundary value problems for canonical domains with subsequent reduction of the boundary value problems for non-canonical domains to the solution of integral equations.

The method was first discussed for strictly mixed problems taking Laplace's equation as an example. It was then applied to contact problems of the theory of elasticity, including the case when the contact zones are unknown in advance.

1. Let $\bar{\Omega}(-\infty<x<\infty;\ 0\leq y\leq\pi)$ (Fig. 1) be a uniform elastic strip with Lamé coefficients λ and μ that lies between two non-deformable half-planes $y<0$ and $y>\pi$. It is assumed that there is no friction on the lines of contact. The upper boundary of the strip is partially detached from the half-plane $y>\pi$ as a result of a vertical displacement of a pair of absolutely rigid inclusions inside the strip.

By virtue of the symmetry of the problem with respect to the axis $x=0$, the displacement vector

$$\mathbf{U}(x, y) = [u_1(x, y),\ u_2(x, y)]^T$$

of the points of the half-strip $\Omega(0\leq x<\infty;\ 0\leq y<\pi)$ is determined by the following system of Lamé equations and the corresponding boundary conditions:

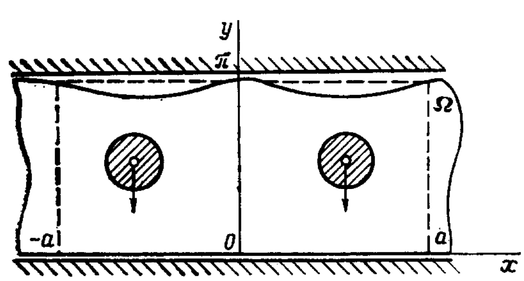

Fig. 1.

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$$LU = 0 \tag{1.1}$$

$$\begin{gathered} u_1 = 0,\ \tau_{xy} = 0 \ \text{for} \ x = 0;\ u_2 = 0,\ \tau_{yx} = 0 \ \text{for} \ y = 0 \\ u_2 = 0 \ \text{for} \ x \geqslant a;\ \sigma_y = 0 \ \text{for} \ x < a,\ \tau_{yx} = 0 \ \text{for} \ y = \pi \\ u_1 = 0,\ u_2 = H = \text{const} \ \text{for} \ (x, y) \in \Gamma \end{gathered} \tag{1.2}$$

(L is the Lamé linear differential operator for the plane problem of elasticity theory and Γ is the contour of the inclusion). The point $x = a$ at which the type of the boundary conditions on the upper edge of the half-strip changes is unknown and is to be determined along with the components of the stress–strain state of the strip.

Problems concerned with the solvability of strictly mixed boundary value problems of elasticity theory for a strip, the differential properties of the solutions, and the behaviour of the solutions both at infinity and at the points at which the type of boundary conditions changes have been investigated in [2]. The solution of problem (1.1), (1.2) has continuous first-order and second-order partial derivatives everywhere in Ω, except, of course, at the points where the type of boundary conditions changes.

We shall solve the problems in two stages. First, we shall consider the corresponding problem with fixed a, and we shall then determine a, e.g. from the continuity conditions

$$\sigma_y(a_-, \pi) = \sigma_y(a_+, \pi) = 0 \tag{1.3}$$

for the σ_y component of the stress tensor at that point.

We will extend the definition of the component u_2 of $\mathbf{U}$ for $y = \pi$ in the interval $[0, a)$ by means of a continuous, triply differentiable function $\varphi(x)$, which leads to the relations

$$\begin{gathered} u_1 = 0,\ \tau_{xy} = 0 \ \text{for} \ x = 0 \\ u_2 = 0,\ \tau_{yx} = 0 \ \text{for} \ y = 0;\ u_2 = \Phi(x),\ \tau_{yx} = 0 \ \text{for} \ y = \pi \\ u_1 = 0,\ u_2 = H = \text{const} \ \text{for} \ (x, y) \in \Gamma \\ \Phi(x) = \varphi(x)\,[1 - \delta_0(x - a)] \end{gathered} \tag{1.4}$$

[$\delta_0(x-a)$ is the Heaviside function].

It is obvious that for the conditions $u_x'(0, y) = 0$ and $u_2(x, \pi) = 0$ to be satisfied for $a \leqslant x < \infty$, it is necessary to set

$$\varphi'(0) = 0,\ \varphi(a) = 0 \tag{1.5}$$

Henceforth we shall express the solution of problem (1.4) for system (1.1) in the form

$$\begin{gathered} \mathbf{U} = \mathbf{V} + \mathbf{S} \qquad \mathbf{V} = \mathbf{V}(x, y) = [v_1(x, y),\ v_2(x, y)]^T \\ \mathbf{S} = \mathbf{S}(x, y) = [-y\Phi'(x)/(2\pi),\ y\Phi(x)/\pi]^T \end{gathered} \tag{1.6}$$

Thus, setting $\varphi'(a) = 0$, we can write

$$L\mathbf{V} = \mathbf{F} \qquad v_1 = 0,\ \tau_{xy} = 0 \ \text{for} \ x = 0 \tag{1.7}$$

$$\begin{gathered} v_2 = 0,\ \tau_{yx} = 0 \ \text{for} \ y = 0,\ y = \pi \\ v_1 = y\Phi(x)/(2\pi),\ v_2 = H - y\Phi(x)/\pi \ \text{for} \ (x, y) \in \Gamma \end{gathered} \tag{1.8}$$

for $\mathbf{V}$, where $\mathbf{F} = \mathbf{F}(x, y)$ is a vector-valued function with components

$$f_1(x, y) = (2\pi)^{-1}(\lambda + 2\mu) y^2 \{\varphi''(x)[1 - \delta_0(x - a)] - \varphi''(a)\delta(x - a)\} - \pi^{-1}\lambda\varphi'(x)[1 - \delta_0(x - a)]$$

$$f_2(x, y) = \pi^{-1}\lambda y \{\varphi''(x)[1 - \delta_0(x - a)]\}$$

[$\delta(x-a)$ is the Dirac delta function].

Using Green's matrix

$$\mathbf{G}(x, y; \xi, \eta) = (\mathbf{G}_{ij}(x, y; \xi, \eta)) \quad (i, j = 1, 2)$$

of the homogeneous boundary value problem corresponding to (1.7) and (1.8), the vector **V** can be represented by the sum of potentials

$$\mathbf{V}(x, y) = \iint_{(\Omega)} \mathbf{G}(x, y; \xi, \eta)\mathbf{F}(\xi, \eta)\, d\Omega(\xi, \eta) + \int_\Gamma \mathbf{G}(x, y; \xi, \eta)\mathbf{N}(\xi, \eta)\, d\Gamma(\xi, \eta) \tag{1.9}$$

where the latter term takes into account the influence of the inclusion.

The density

$$\mathbf{N}(x, y) = [\nu_1(x, y), \nu_2(x, y)]^T$$

of the contour potential is a vector formed by unknown integrable functions on Γ. The expressions for the elements $\mathbf{G}_{ij}$ of Green's matrix have been obtained in [3].

Integrating the first term on the right-hand side of (1.9) with respect to η, we can, in accordance with (1.6), represent the components of **U** in the form

$$\begin{aligned} u_1(x, y) = &-\int_0^\infty \Big\{\frac{\lambda}{\pi} g_{11}{}^0\varphi'(\xi)[1 - \delta_0(\xi - a)] - \Big[\frac{\pi\beta}{6} g_{11}{}^0 + \\ &+ \frac{2\beta}{\pi}\sum \frac{(-1)^k}{k^2} g_{11}{}^n \cos ky\Big]\Big\{\varphi'''(\xi)[1 - \delta_0(\xi - a)] - \varphi''(a)\delta(\xi - a)\Big\} + \\ &+ \frac{2\lambda}{\pi}\sum \frac{(-1)^k}{k} g_{12}{}^n \cos ky\, \varphi''(\xi)[1 - \delta_0(\xi - a)]\Big\} d\xi + I_1 - (2\pi)^{-1}\Phi^1(x) y^2 \end{aligned} \tag{1.10}$$

$$\begin{aligned} u_2(x, y) = &-\int_0^\infty \Big\{-\frac{2\beta}{\pi}\sum \frac{(-1)^k}{k^2} g_{21}{}^n \sin ky \times \\ &\times \{\varphi'''(\xi)[1 - \delta_0(\xi - a)] - \varphi''(a)\delta(\xi - a)\} + \\ &+ \frac{2\lambda}{\pi}\sum \frac{(-1)^k}{k} g_{22}{}^n \sin ky\varphi''(\xi)[1 - \delta_0(\xi - a)]\Big\} d\xi + I_2 \end{aligned}$$

$$I_i = \int_\Gamma [G_{i1}(x, y; \xi, \eta)\nu_1(\xi, \eta) + G_{i2}(x, y; \xi, \eta)\nu_2(\xi, \eta)]\, d\Gamma(\xi, \eta), \quad i = 1, 2$$

Here $g_{ij}{}^n = g_{ij}{}^n(x, \xi)$ are the coefficients of the trigonometric expansions of the elements $\mathbf{G}_{ij}$ of Green's matrix [3], $\beta = \lambda + 2\mu$, and the sums are taken with respect to k from $k = 0$ to $k = \infty$. If one applies the integration by parts to those terms in (1.10) that contain $\varphi'(\xi)$ and $\varphi'''(\xi)$ and takes the sum of those trigonometric series for which this can be done, one can obtain the expression

$$\mathbf{U}(x, y) = \int_0^a \mathbf{R}(x, y, \xi, a)\varphi''(\xi)\, d\xi + \int_\Gamma \mathbf{G}(x, y; \xi, \eta)\mathbf{N}(\xi, \eta)\, d\Gamma(\xi, \eta)$$

$$\mathbf{R} = \mathbf{R}(x, y, \xi, a) = \left\| \begin{matrix} r_1(x, y, \xi, a) \\ r_2(x, y, \xi, a) \end{matrix} \right\| \tag{1.11}$$

$$r_1(x, y, \xi, a) = T_1(x, y, \xi) + T_1(x, y, -\xi) + T_2(x, \xi) + T_3(x, y, \xi, a)$$
$$r_2(x, y, \xi, a) = T_4(x, y, \xi) + T_4(x, y, -\xi) + \pi^{-1}K_1(x, \xi, a)\, y$$
$$T_1(x, y, \xi) = \frac{1}{2\pi}\left\{\frac{\alpha}{\beta}(x+\xi)\left[|x+\xi| - \ln(2(\operatorname{ch}|x+\xi| + \cos y))\right] + \right.$$
$$\left. + 2\operatorname{sign}(x+\xi)\sum \frac{(-1)^k \cos ky}{k^2} e^{-k|x+\xi|}\right\};$$
$$T_2(x, \xi) = \begin{cases} \pi/6 - \lambda\xi^2/(2\pi\beta), & \xi \leqslant x \\ 0, & \xi > x \end{cases}$$
$$\alpha = \lambda + \mu$$
$$T_3(x, y, \xi, a) = \begin{cases} T_5(x, y, \xi) - \lambda x K_1(x, \xi, a)/(\pi\beta), & x < a \\ \lambda a^2/(2\pi\beta), & x > a \end{cases}$$
$$T_4(x, y, \xi) = -\frac{1}{\pi\beta}\left[\alpha|x+\xi| \operatorname{arctg}\frac{\sin y}{\cos y + e^{|x+\xi|}} + \right.$$
$$\left. + \mu \sum \frac{(-1)^k \sin ky}{k^2} e^{-k|x+\xi|}\right]$$
$$T_5(x, y, \xi) = \begin{cases} (\lambda x^2/\beta - y^2)\, 2\pi, & x \geqslant \xi; \\ 0, & x < \xi; \end{cases} \qquad K_1(x, \xi, a) = \begin{cases} x - a, & x \geqslant \xi \\ \xi - a, & x < \xi \end{cases}$$

Next we use the substitution $\varphi''(x) = \psi(x)$ and we require that the representation (1.11) complies with the last three boundary conditions in (1.2). Then we obtain the system of integral equations

$$\int_0^a [\lambda r_{1x}(x, \pi, \xi, a) + \beta r_{2y}(x, \pi, \xi, a)]\,\psi(\xi)\,d\xi + J_1 + J_2 = 0 \tag{1.12}$$
$$\left(J_i = \int_\Gamma \left\{\left[\lambda \frac{\partial G_{1i}}{\partial x}(x, \pi, \xi, \eta) + \beta \frac{\partial G_{2i}}{\partial y}(x, \pi, \xi, \eta)\right] \nu_i(\xi, \eta)\, d\Gamma(\xi, \eta)\right)\right\}$$
$$\int_0^a r_i(x, y, \xi, a)\,\psi(\xi)\,d\xi + I_i = H\delta_{i2}, (x, y) \in \Gamma, \quad i = 1, 2$$
$$\left(\delta_{i2} = \begin{cases} 0, & i = 1 \\ 1, & i = 2 \end{cases}\right)$$

with respect to the unknown functions $\psi(x)$, $\nu_i(x, y)$. Here we use the notation introduced above and we also introduce the following new notation

$$r_{1x}(x, y, \xi, a) = T_{1x}(x, y, \xi) + T_{1x}(x, y, -\xi) - \lambda(\pi\beta)^{-1} K_1(x, \xi, a)$$
$$r_{2y}(x, y, \xi, a) = T_{2y}(x, y, \xi) + T_{2y}(x, y, -\xi) + \pi^{-1}K_1(x, \xi, a)$$
$$T_{1x}(x, y, \xi) = \frac{1}{2\pi\beta}\left\{\alpha(x+\xi)\left[1 - \frac{\operatorname{sh}|x+\xi|}{\operatorname{ch}|x+\xi| + \cos y}\right] - \mu M(x, y, \xi)\right\}$$
$$T_{2y}(x, y, \xi) = \frac{1}{2\pi\beta}\left\{\alpha|x+\xi| \frac{1 + e^{|x+\xi|}\cos y}{1 + 2\cos y\, e^{|x+\xi|} + e^{2|x+\xi|}} - \mu M(x, y, \xi)\right\}$$
$$M(x, y, \xi) = |x+\xi| - \ln[2(\operatorname{ch}|x+\xi| + \cos y)]$$

The kernels of the equations of system (1.12) have weak singularities, which enables one to apply the natural regularization method based on separating the singularities in a kernel by the use of the Krylov–Bogolyubov method [4] when solving (1.12). In this case the step h of the partition of integration domain serves as the regularization parameter and can be chosen by means of a numerical experiment.

We introduce a parameterization of the points (x, y) of Γ with the aid of the relations

$$x = x(t),\ y = y(t),\ \xi = x(\tau),\ \eta = y(\tau),\ t_1 \leqslant t,\ \tau \leqslant t_2$$

Then, dividing $[0, a)$ and Γ into p and m equal parts, respectively, we can approximate (1.12) by a regular system of linear algebraic equations with respect to the approximate values z_j of the unknown functions $\psi(x)$, $\nu_1(x, y)$, and $\nu_2(x, y)$ at the middle points of the intervals of the partitions:

$$\sum_{j=1}^{p+2m} B_{ij} z_j = q_i \tag{1.13}$$

$$z_j = \begin{cases} \psi(\xi_j); & j = 1, 2, \ldots p \\ \nu_1(x(\tau_{j-p}),\ y(\tau_{j-p})), & j = p+1,\ p+2, \ldots, p+m \\ \nu_2(x(\tau_{j-p-m}),\ y(\tau_{j-p-m})), & j = p+m+1, \\ p+m+2, \ldots, p+2m \end{cases}$$

$$q_i = \begin{cases} 0, & i = 1, 2, \ldots,\ p+m \\ H, & i = p+m+1,\ p+m+2, \ldots,\ p+2m \end{cases}$$

The computation of the diagonal elements B_{ij} of the matrix of system (1.13) involves improper integrals of the form

$$\int_{x_i-h/2}^{x_i} \ln(1 - \exp(-|x_i - \xi_j|))\, d\xi$$

which we can evaluate by approximating the argument of the logarithm by the linear part of its Taylor expansion.

By computing the characteristics of the stress–strain state of the half-strip Ω using the approximate solution of (1.12) and determining, with the aid of (1.3), the coordinate $x = a$ of the point of detachment of the half-strip from the support, we can finally obtain the picture of the stress–strain state of the half-strip under investigation.

2. Let us state some results of the computations based on the method described.

Figure 2 shows the displacement of the upper boundary of the elastic strip loaded by two symmetrically applied concentrated mass forces normal to the horizontal boundaries of the strip for various locations of the points of application of the forces. It is of interest to note that the mutual influence of the forces virtually disappears as the two points of application of the forces move away from each other in the horizontal direction so that the distance between them becomes at least twice as large as the width of the strip (we recall that the y axis in this formulation of the problem is an axis of symmetry for the field under investigation, which means that

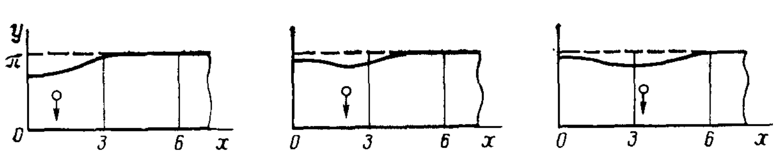

FIG. 2.

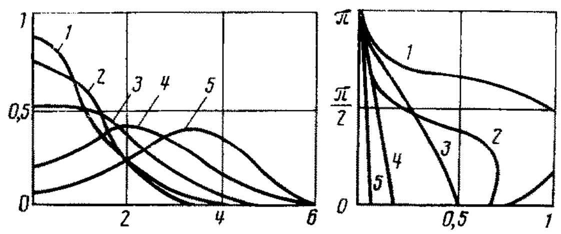

FIG. 3.

an elastic strip with two forces acting upon it is considered). In other words, for a single force, the distance between the points of detachment of the upper edge of the strip and the point of application of the force is approximately equal to the width of the strip.

In Fig. 3(a) we show diagrams of the σ_y component of the stress tensor on the lower edge of the strip for various values x_0 of the coordinate of the point of application of the force (curves 1–5 correspond to $x_0 = 0; 0.5; 1; 2; 3$). The following fact is worthy of note. In the case where the forces are sufficiently far away from the y axis (curves 4 and 5), if the maximum of the curve is attained at the point below the point of application of the force, then the maximum moves towards the axis of symmetry as the forces move closer to each other. The diagrams of the σ_y component of the stress tensor on the line $x = 0$ are shown in Fig. 3(b) for the same configurations of the points of application of the forces. The virtual absence of stress of $x_0 = 3$ (curve 5) provides yet another confirmation of the fact that the mutual influence of the forces is significant to some degree only if the distance between them is less than twice the width of the strip.

The stress and displacement fields in an elastic strip under the conditions modelled by problem (1.2) are shown in Fig. 4. The deformed coordinate mesh of the strip is shown in the right upper corner. The left upper corner, the left lower corner, and the right lower corner present the stress fields τ_{max}, σ_{max}, and σ_{min}, respectively. An analysis of these fields enables us to conclude that, in particular, the zone lying directly below the inclusion is subjected to the greatest strain. Obviously, this is where one should expect zones of plasticity and crack formation to appear in the first place under real conditions. Strong compressive stress is also present in the vicinity of the origin of the system of coordinates. We also note that, although the stress level above an inclusion is much lower than that under consideration (which is, obviously, natural), the form of its distribution is interesting because of the presence of zones of compressive as well as tensile stress.

It follows that the foramlism of Green matrices is an efficient method for solving problems that involve elastic bodies detached from a rigid support.

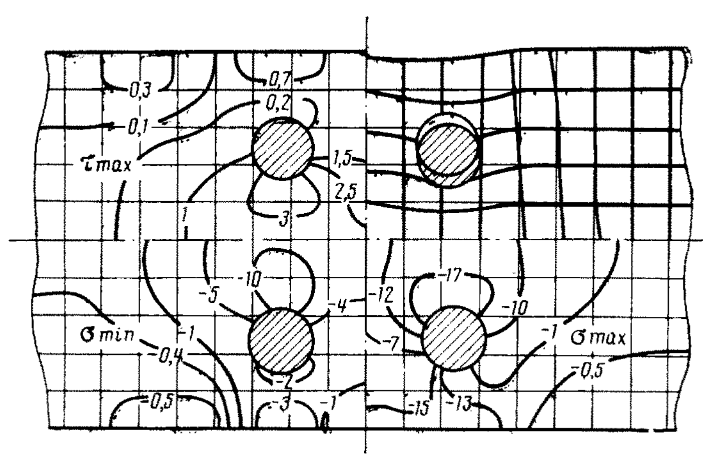

FIG. 4.

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ASYMPTOTIC BEHAVIOUR OF THE SOLUTIONS OF BOUNDARY VALUE PROBLEMS OF POTENTIAL THEORY AND THE THEORY OF ELASTICITY IN THE VICINITY OF CONICAL POINTS ON THE BOUNDARY†

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One-dimensional integral equations of the second kind are constructed in order to determine the asymptotic behaviour of the solutions of the boundary value problems of potential theory and the theory of elasticity in the vicinity of conical points on the boundary surface. An algorithm for solving the problems is described and computational results for model examples are presented.

As a result of studying the solutions of the boundary value problems for elliptic equations in the vicinity of conical points on the boundary surface it has been shown [1] that, in any domain of this type, the solution can be represented as the sum of an infinitely differentiable function and an asymptotic series, each term of which is a solution of the homogeneous boundary value problem for an infinite cone formed by the half-lines tangent at the conical point. The solutions in question (eigenfunctions) are determined only by the local structure of the conical surface and the type of boundary conditions. Naturally, the coefficients multiplying the solutions depend on the general configuration of the domain and the values of the boundary conditions. It is obvious that the

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